MULTI-VALUED CONTRACTION MAPPINGS IN GENERALIZED METRIC SPACES

BY
H. COVITZ AND S. B. NADLER, JR.

ABSTRACT

Several fixed point theorems for multi-valued global and local contraction mappings are proved. These results concerning contractions are then applied to obtain a fixed point theorem for a certain type of single-valued locally expansive mapping.

1. **Introduction.** The purpose of this paper is to prove a general fixed point theorem for multi-valued contraction mappings which is used to put the results in [7] in a more general and suitable setting. This enables us to answer (see Corollary 4 below), in more generality than was asked for, a question posed at the end of section 3 of [7] and to extend significantly theorem 7 in [7] concerning the existence of fixed points of locally expansive mappings (see Theorem 2 below).

The main theorem of this paper, Theorem 1, is a result about local contractions and is used to obtain results about global contractions. This is a somewhat different approach than that taken in [2]. In [2] the same iterative type of proof had to be redone for each of the theorems. Our approach in this paper has the feature that only Theorem 1 is proved with an iterative method; the other results, though they are about iterates of functions, are obtained from Theorem 1 as corollaries with "soft" proofs. We remark that theorems 5 and 6 of [7] can be proved in the general setting of Corollaries 3 and 4 of this paper in a direct fashion. However, since Theorem 1 below is a unifying tool which simultaneously extends results in [1], [2], and [5] as well as results in [7], we have chosen to obtain our results using Theorem 1 below.

2. **Basic concepts.** A *generalized metric space* (see [5], p. 541) is a pair (X,d) where X is a (nonempty) set and $d\colon X \times X \to [0,\infty]$ satisfies all the properties of being a metric for X except that d may have "infinite values". A general-

Received June 12, 1969

ized metric space (X,d) is said to be *complete* iff every d-Cauchy sequence in X (i.e., $\{x_n\}_{n=1}^{\infty}$ is a d-Cauchy sequence in X iff $\lim_{n,m\to\infty} d(x_n,x_m) = 0$) is d-convergent to a point in X. We remark that a generalized metric space can be "remetrized" with a genuine metric by taking the minimum of the generalized metric and the real number one. The "remetrization" preserves the topology but changes the Lipschitz structure of the space. Since we shall be dealing with contraction mappings in this paper, the generalized metric space structure cannot be replaced by ametric space structure.

If (X,d) is a generalized metric space, then

(a) $CL(X) = \{C \mid C \text{ is a nonempty closed subset of } X\}$,

(b) $N(\varepsilon,C) = \{x\in X \mid d(x.c) < \varepsilon \text{ for some } c\in C\}$ if $\varepsilon > 0$ and $C\in CL(X)$, and

(c) $$H(A,B) = \begin{cases} \inf\{\varepsilon > 0 \mid A\subset N(\varepsilon,B) \text{ and } B\subset N(\varepsilon,A)\}, & \text{if the infimum exists} \\ \infty & \text{, otherwise} \end{cases}$$

if $A,B\in CL(X)$.

The pair $(CL(X),H)$ is a generalized metric space and H is called the *generalized Hausdorff distance induced by d* (in general H depends on d but we shall not notate this except where confusion may arise). A function $F: X \to CL(X)$ is called a *multi-valued contraction mapping* (abbreviated *m.v.c.m*) iff there exists a fixed real number $\lambda < 1$ such that $H(F(x),F(y)) \leqq \lambda\cdot d(x,y)$ for all $x,y\in X$ such that $d(x,y)<\infty$. A function $F: X \to CL(X)$ is called an *(ε,λ)-uniformly locally contractive multi-valued mapping* (where $\varepsilon > 0$ and $0 \leqq \lambda < 1$) iff $H(F(x),F(y)) \leqq \lambda\cdot d(x,y)$ for all $x,y\in X$ such that $d(x,y)<\varepsilon$. This definition is an extension of Edelstein's definition for single-valued uniformly locally contractive mappings. The notion of a multi-valued contraction mapping does not seem to have appeared prior to the announcement in [6].

Let (X,d) be a generalized metric space, let $x_0\in X$, and let $F: X\to CL(X)$ be a function. A sequence $\{x_i\}_{i=1}^{\infty}$ of points of X is said to be an *iterative sequence of F at x_0* iff $x_i\in F(x_{i-1})$ for each $i = 1,2,\cdots$.

A point $p\in X$ is a *fixed point* of a function $F: X\to CL(X)$ iff $p\in F(p)$.

3. Fixed point theorems.

THEOREM 1. *Let (X,d) be a generalized complete metric space and let $x_0\in X$. If $F: X\to CL(X)$ is an (ε,λ)-uniformly locally contractive multi-valued mapping, then the following alternative holds: either*

(1.1) *for each iterative sequence $\{x_i\}_{i=1}^{\infty}$ of F at x_0, $d(x_{i-1},x_i)\geqq\varepsilon$ for each $i = 1,2,\cdots$, or*

(1.2) *there exists an iterative sequence* $\{x_i\}_{i=1}^{\infty}$ *of* F *at* x_0 *such that* $\{x_i\}_{i=1}^{\infty}$ *converges to a fixed point of* F.

PROOF. Suppose (1.1) does not hold. Then there is a choice (*) of $x_1 \in F(x_0)$, $x_2 \in F(x_1), \cdots$, and $x_N \in F(x_{N-1})$ such that $d(x_{N-1}, x_N) < \varepsilon$ for some fixed integer $N \geqq 1$. This implies $H(F(x_{N-1}), F(x_N)) \leqq \lambda \cdot d(x_{N-1}, x_N) < \lambda \cdot \varepsilon$. Therefore, since $x_N \in F(x_{N-1})$, there exists $x_{N+1} \in F(x_N)$ such that $d(x_N, x_{N+1}) < \lambda \cdot \varepsilon \ (< \varepsilon)$. Now $H(F(x_N),\ F(x_{N+1})) \leqq \lambda \cdot d(x_N, x_{N+1}) < \lambda^2 \cdot \varepsilon$ and hence, since $x_{N+1} \in F(x_N)$, there exists $x_{N+2} \in F(x_{N+1})$ such that $d(x_{N+1}, x_{N+2}) < \lambda^2 \cdot \varepsilon$. Continuing in this fashion we produce a sequence $\{x_{N+1}\}_{i=1}^{\infty}$ of points of X such that $x_{N+i+1} \in F(x_{N+i})$ and $d(x_{N+i}, x_{N+i+1}) < \lambda^{i+1} \cdot \varepsilon$ for all $i \geqq 1$. It follows that the sequence $\{x_i\}_{i=1}^{\infty}$ is a Cauchy sequence which, by the completeness of (X,d), converges to some point $p \in X$. Hence, the sequence $\{F(x_i)\}_{i=1}^{\infty}$ converges to $F(p)$ and, since $x_{i+1} \in F(x_i)$ for all i and $F(p)$ is closed, $p \in F(p)$. This proves F has a fixed point. Furthermore, the sequence $\{x_i\}_{i=1}^{\infty}$ satisfies the conditions in (1.2) of the alternative.

COROLLARY 1. *Let* (X,d) *be a generalized complete metric space and let* $x_0 \in X$. *If* $F: X \to CL(X)$ *is a m.v.c.m., then the following alternative holds: either*

(1) *for each iterative sequence* $\{x_i\}_{i=1}^{\infty}$ *of* F *at* x_0, $d(x_{i-1}, x_i) = \infty$ *for each* $i = 1, 2, \cdots$,

or

(2) *there exists an iterative sequence* $\{x\}_{i=1}^{\infty}$ *of* F *at* x_0 *such that* $\{x\}_{i=1}^{\infty}$ *converges to a fixed point of* F.

PROOF. Suppose (1) does not hold. Then there is an iterative sequence $\{x\}_{i=1}^{\infty}$ of F at x_0 such that $d(x_{N-1}, x_N) < \infty$ for some fixed integer $N \geqq 1$. Let $\varepsilon < \infty$ be given such that $d(x_{N-1}, x_N) < \varepsilon$. Clearly F is an (ε, λ)-uniformly locally contractive multi-valued mapping which, since (1.1) of Theorem 1 is violated by the iterative sequence above, must satisfy (1.2) of Theorem 1. But this is (2) of the alternative in this corollary.

We say that a generalized metric space (X,d) is *ε-chainable* (where $\varepsilon > 0$ is a fixed real number) iff given $x, y \in X$ such that $d(x,y) < \infty$ there is an ε-chain from x to y (that is, a finite set of points $z_0 = x,\ z_1 \cdots, z_n = y$ such that $d(z_{i-1}, z_i) < \varepsilon$ for all $i = 1, 2, \cdots, n$). The proof of the next theorem is similar to the proof of theorem 6 of [7] (compare with the remark at the end of section 3 of [7]).

COROLLARY 2. *Let (X,d) be a complete ε-chainable generalized metric space and let $x_0 \in X$. If $F : x \to CL(X)$ is an (ε,λ)-uniformly locally contractive multi-valued mapping, then the following alternative holds: either*

(1) *for each iterative sequence $\{x\}_{i=1}^{\infty}$ of F at x_0, $d(x_{i-1},x_i) = \infty$ for each $i = 1,2,\cdots$;*

or

(2) *there exists an iterative sequence $\{x_i\}_{i=1}^{\infty}$ of F at x_0 such that $\{x_i\}_{i=1}^{\infty}$ converges to a fixed point of F.*

PROOF. We define a new generalized metric $d_\varepsilon: X \times X \to [0,\infty]$ by $d_\varepsilon(x,y) = \inf\{ \sum_{i=1}^{n} d(z_{i-1},z_i) \mid z_0 = x, z_1,\cdots,z_n = y$ is an ε-chain from x to $y\}$ if $d(x,y) < \infty$ and $d_\varepsilon(x,y) = \infty$ if $d(x,y) = \infty$. It is easy to verify that (X,d_ε) is a generalized complete metric space. Let H_ε be the generalized Hausdorff metric on $CL(X)$ obtained from d_ε (note that, since $d(x,y) < \varepsilon$ implies $d_\varepsilon(x,y) = d(x,y)$, $CL(X)$ with respect to d is the same set as $CL(X)$ with respect to d_ε). We now show that F is a m.v.c.m. with respect to d_ε and H_ε. First note that if $A,B \in CL(X)$ and $H(A,B) < \varepsilon$, then $H_\varepsilon(A,B) = H(A,B)$ (where H is the generalized Hausdorff metric obtained from d). Now let $x,y \in X$ such that $d(x,y) < \infty$. Let $z_0 = x, z_1,\cdots,z_n = y$ be an ε-chain from x to y. Then

$$H_\varepsilon(F(x),F(y)) \leqq \sum_{i=1}^{n} H_\varepsilon(F(z_{i-1}),F(z_i)) = \sum_{i=1}^{n} H(F(z_{i-1},F(z_i)) \leqq$$

$$\sum_{i=1}^{n} \lambda \cdot d(z_{i-1},z_i) = \lambda \cdot \sum_{i=1}^{n} (z_{i-1},z_i),$$

i.e.,

$$H_\varepsilon(F(x), F(y)) \leqq \lambda \cdot \sum_{i=1}^{n} d(z_{i-1},z_i).$$

Since $z_0 = x, z_1,\cdots,z_n = y$ was an arbitrary ε-chain from x to y, it follows that $H_\varepsilon(F(x),F(y) \leqq \lambda d_\varepsilon(x,y)$. This proves F is a m.v.c.m. with respect to d_ε and H_ε. Now, since d_ε is equivalent to d, Corollary 1 may be applied to complete the proof of this corollary.

The next corollary follows immediately from Corollary 1 above and extends theorem 5 of [7]. In particular, theorem 5 of [7] required that the multi-valued contraction mapping map into $CB(X) = \{C \mid C$ is a nonempty, closed, and bounded subset of $X\}$. The boundedness restriction was imposed so that the hyperspace was a genuine metric space. It is not necessary that the hyperspace be metric and, in fact, the boundedness of point images was not used in the proof of theorem 5 of [7].

COROLLARY 3. *Let* (X,d) *be a complete metric space and let* $x_0 \in X$. *If* $F: X \to CL(X)$ *is a m.v.c.m., then there exists an iterative sequence* $\{x_i\}_{i=1}^{\infty}$ *of* F *at* x_0 *such that* $\{x_i\}_{i=1}^{\infty}$ *converges to a fixed point of* F.

The following corollary is an immediate consequence of Corollary 2 above. It is a substantial extension of theorem 6 of [7] which states that an (ε,λ)-uniformly locally contractive multi-valued mapping F on a complete ε-chainable metric space has a fixed point if each point image $F(x)$ is nonempty and compact.

COROLLARY 4. *Let* (X,d) *be a complete* ε*-chainable metric space and let* $x_0 \in X$. *If* $F: X \to CL(X)$ *is an* (ε,λ)*-uniformly locally contractive multi-valued mapping, then there exists an iterative sequence* $\{x_i\}_{i=1}^{\infty}$ *of* F *at* x_0 *such that* $\{x_i\}_{i=1}^{\infty}$ *converges to a fixed point of* F.

Let (X,d) be a metric space. A single-valued mapping f is said to be (ε,λ)-*uniformly locally expansive* (where $\varepsilon > 0$ and $\lambda > 1$) provided that, if x and y are in the domain of f and $d(x,y) < \varepsilon$, then $d(f(x),f(y)) \geqq \lambda \cdot d(x,y)$.

Theorem 6 of [7] was used to obtain, via the inverse function, fixed point theorems for uniformly locally expansive single-valued mappings which are not necessarily one-to-one (see theorem 7 of [7]). These results corrected and extended a result of Edelstein [3]. However, due to the compactness requirement on point images in theorem 6 of [7], a compactness requirement was needed on the inverse images of points in theorem 7 of [7]. Corollary 4 above enables us to eliminate this compactness requirement and prove the following extension of theorem 7 of [7].

THEOREM 2. *Let* (X,d) *be a complete* ε*-chainable (respectively, well-chained) metric space, let* A *be a nonempty subset of* X, *and let* $f: A \to X$ *be an* (ε,λ)*-uniformly locally expansive (continuous) mapping of* A *onto* X. *If* $f^{-1}(x)$ *is closed* in X *for each* $x \in X$ *and* $f^{-1}: X \to CL(X)$ *is* ε*-nonexpansive (respectively, uniformly* ε*-continuous), then* f *has a fixed point.*

To see that some metric type of restriction even stronger than uniform continuity must be placed on f^{-1}, the reader is referred to example 3 of [7].

Theorem 2 above makes theorem 8 of [7] superfluous (see the remark at the end of section 3 of [7]).

4. **Remarks 1.** In the proof of Theorem 1 we continued the finite choice (*)

and produced an iterative sequence not satisfyng (1.1) which, because of the method by which it was chosen, satisfied (1.2). In general, an arbitrary iterative sequence $\{x_i\}_{i=1}^{\infty}$ not satisfying (1.1) may not converge (much less satisfy (1.2)). As an illustration of this let (X,d) be the generalized metric space consisting of a two element set $X = \{a,b\}$ and the "generalized distance function" d given by $d(a,b) = \infty = d(b,a)$ and $d(a,a) = 0 = d(b,b)$. Define $F: X \to CL(X)$ by $F(x) = X$ for each $x \in X$. Let $X_0 = a$. The iterative sequence $\{x_i\}_{i=1}^{\infty}$, where $x_1 = a$, $x_2 = b$, $x_3 = a$, $x_4 = b, \cdots$, does not satisfy (1.1) because $d(x_0,x_1) = 0$ (since F is a constant mapping, ε may be taken as any strictly positive real number) and clearly does not satisfy (1.2). If f is a single-valued mapping satisfying the hypotheses of any of the results above then, since the one-element sets are isometrically embedded in $CL(X)$, the set-valued mapping $F(x) = \{f(x)\}$ for all $x \in X$ also satisfies these hypotheses. In this case the alternative statements no longer involve "choices", but they are concerned with the unique iterative sequence (in set brackets) of f at x_0. From these observations it follows that each result in this section gives directly the corresponding result for single-valued mappings. Thus, for example, Theorem 1 above gives the theorem in section 3 of [2], Corollary 3 gives the contraction mapping principle of Banach [1], etc.

2. The results in section 3 are closely related in that one can be obtained from the other. The technique for obtaining multi-valued theorems for generalized metric spaces by using multi-valued theorems for metric spaces is especially interesting. A method for doing this for single-valued mappings was suggested and carried out by Jung in [4]. We indicate briefly how to extend Jung's technique to the multi-valued case by sketching a proof of Corollary 1 of the previous section from Corollary 3 of the previous section. Let us assume Corollary 3. Suppose (1) of Corollary 1 does not hold and let $x_1 \in F(x_0), x_2 \in F(x_1), \cdots, x_N \in F(x_{N-1})$ be a choice such that $d(x_{N-1},x_N) < \infty$ for some fixed integer $N \geqq 1$. Let $[x_N] = \{x \in X \mid d(x,x_N) < \infty\}$. Since (X,d) is a generalized complete metric space, $[x_N]$ is a complete metric space [4]. However, F may not map $[x_N]$ into $CL([x_N])$. Define G on $[x_N]$ by $G(x) = F(x) \cap [x_N]$ for each $x \in [x_N]$. It can be shown that $G(x) \neq \phi$ for each $x \in [x_N]$ and, hence, that G is a function from $[x_N]$ into $CL([x_N])$. Also, though in general the intersection of a multi-valued contraction mapping with a fixed set may not be continuous (even if the intersection is nonempty), it can be shown that G is a multi-valued contraction mapping. This essentially follows from the fact that F is a m.v.c.m. and the points of

$X - [x_N]$ are infinitely far from the points of $[x_N]$. We may now apply Corollary 3 to G and, since $G(x) \subset F(x)$ for all $x \in X$, the result follows for F.

REFERENCES

1. S. Banach, *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales*, Fund. Math. **3** (1922), 133–181.

2. J. B. Diaz and B. Margolis, *A fixed point theorem of the alternative for contractions on a generalized complete metric space*, Bull. Amer. Math. Soc., **74** (1968), 305–309.

3. M. Edelstein, *An extension of Banach's contraction principle*, Proc. Amer. Math. Soc., **12** (1961), 7–10.

4. C. F. K. Jung, *On generalized complete metric spaces*, Bull. Amer. Math. Soc., **75** (1969), 113–116.

5. W. A. J. Luxemburg, *On the convergence of successive approximations in the theory of ordinary differential equations, II*, Koninkl. Nederl. Akademie van Wetenschappen, Amsterdam, Proc. Ser. A(**5**), 61, and Indag. Math. (5), 20 (1958), 540–546.

6. S. B. Nadler, Jr., *Multi-valued contraction mappings*, Notices Amer. Math. Soc., **14** (1967), 930.

7. S. B. Nadler, Jr., *Multi-valued contraction mappings*, Pacific J. Math., **30** (1969), 415–487.

STATE UNIVERSITY OF NEW YORK AT BUFFALO